Thermodynamics of UNNS: Entropy, Energy, and Non-equilibrium Recursion

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Abstract

We extend the Unbounded Nested Number Sequence (UNNS) substrate into a thermodynamic framework. By defining energy, entropy, and temperature of recursion, we establish analogies to statistical mechanics, information theory, and complexity science. We construct ensembles of nests, develop energy functionals based on echo residues and spectral constants, and analyze entropy production under operator-driven dynamics. We further study phase transitions, fluctuation theorems, entropy rates, and the role of repair operators as thermodynamic work. Finally, we propose simulation strategies and discuss implications for non-equilibrium systems.

Contents

1 Introduction

The UNNS substrate views recursion as a generative substrate for mathematics and physics. Earlier work introduced operators (inlaying, inletting, repair, trans-sentifying) and constants (universal fixed points, echo residues, paradox index). Here, we propose a thermodynamic extension, treating recursive nests as microstates and defining ensembles, entropy, energy, and temperature.

This approach aims to:

- provide metrics for stability and complexity of recursion;
- bridge UNNS with statistical mechanics and information theory;
- establish a foundation for non-equilibrium analysis of operator-driven recursion.

2 Ensembles of Nests

2.1 Nests as microstates

Let $\mathcal{N}_k$ be the set of nests truncated to depth k. Each $N \in \mathcal{N}_k$ is described by coefficients $\mathbf{a} = (a_0, \ldots, a_k)$, echo residues $\{e_n\}$, and spectral constants $\sigma(N)$.

Definition 2.1 (Energy functional)**.** *An* energy functional *is a map*

$$E : \mathcal{N}_k \to \mathbb{R}_{\geq 0}$$

measuring instability, misfit, or complexity of a nest.

Examples include:

$$E_{\text{echo}}(N) = \sum_{n=0}^{k} w_n |e_n|^2,$$
$$E_{\text{spec}}(N) = \operatorname{dist}\bigl(\sigma(N), \mathcal{O}_K\bigr)^2,$$
$$E_{\text{comp}}(N) = \ell(N),$$

where $\ell(N)$ is description length and $\mathcal{O}_K$ an algebraic integer ring.

2.2 Canonical ensemble

Definition 2.2 (UNNS canonical ensemble)**.** *For $\beta > 0$, define*

$$p_\beta(N) = \frac{1}{Z(\beta)} e^{-\beta E(N)}, \qquad Z(\beta) = \sum_{N \in \mathcal{N}_k} e^{-\beta E(N)}.$$

3 Thermodynamic Quantities

Definition 3.1 (Average energy)**.**

$$\langle E \rangle_\beta = \sum_N p_\beta(N) E(N).$$

Definition 3.2 (Entropy)**.**

$$S(\beta) = -\sum_N p_\beta(N) \log p_\beta(N) = \beta \langle E \rangle_\beta + \log Z(\beta).$$

Definition 3.3 (Free energy)**.**

$$F(\beta) = -\frac{1}{\beta} \log Z(\beta).$$

Proposition 3.1 (Thermodynamic relation)**.** *Entropy satisfies*

$$\frac{\partial S}{\partial \langle E \rangle} = \frac{1}{T},$$

where $T = 1/\beta$ is the UNNS temperature.

4 Non-equilibrium Operator Dynamics

Recursive operators act as stochastic or deterministic maps $N \mapsto N'$.

Definition 4.1 (Entropy production rate)**.** *Let $P_t(N)$ evolve under a master equation with transition rates $W(N \to N')$. Then*

$$\sigma(t) = \frac{d}{dt} S(t) + \sum_{N,N'} J_{N\to N'}(t) \ln \frac{W(N \to N')}{W(N' \to N)},$$

with $J_{N\to N'}(t) = P_t(N)W(N \to N') - P_t(N')W(N' \to N)$.

Lemma 4.1 (Second law for UNNS)**.** *For any operator evolution,*

$$\Delta S_{sys} + \Delta S_{env} \geq 0.$$

5 Phase Transitions in Recursion

5.1 Order parameters

Candidate order parameters include:

- average echo-energy $\langle E_{\text{echo}} \rangle$,
- spectral coherence $|\langle \sigma(N) \rangle|$,
- per-layer growth exponent λ.

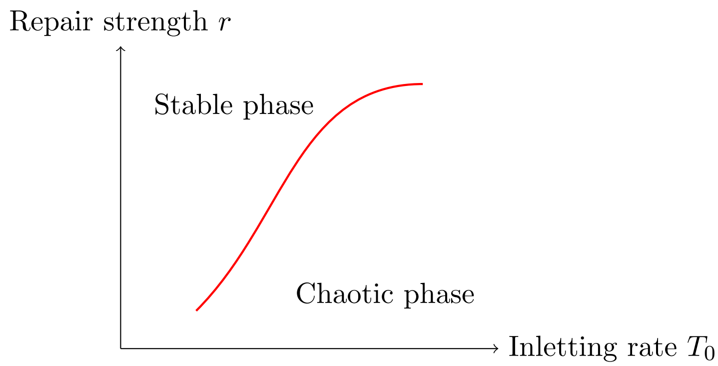
5.2 Control parameters

Control parameters include inletting rate T_0, repair strength r, and noise amplitude η.

Theorem 5.1 (Critical regime)**.** *There exist curves in parameter space (T_0, r, η) along which susceptibilities diverge:*

$$C(\beta) = \beta^2(\langle E^2 \rangle - \langle E \rangle^2) \to \infty.$$

5.3 Phase diagram



6 Fluctuation Theorems

Proposition 6.1 (Jarzynski equality for UNNS)**.** *Let W be operator work and ΔF free energy difference. Then*

$$\langle e^{-\beta W} \rangle = e^{-\beta \Delta F}.$$

Proposition 6.2 (Crooks relation)**.**

$$\frac{P_F(W)}{P_R(-W)} = e^{\beta(W-\Delta F)}.$$

7 Entropy Rates and Complexity

Definition 7.1 (Entropy rate)**.** *The UNNS entropy rate is*

$$h_{UNNS} = \lim_{k \to \infty} \frac{1}{k} H(\textit{prefix of depth } k).$$

Remark 7.1. *h_{UNNS} measures information growth per layer, analogous to Shannon entropy rate.*

Definition 7.2 (Spectral entropy)**.** *Entropy of the distribution of phases of $\sigma(N)$.*

8 Repair Operators as Work

Repair operators reduce system entropy but require work.

Definition 8.1 (Repair work)**.** *If $R(N)$ is the repaired state,*

$$W_{repair} = d(N, R(N)),$$

where d is a metric on nests.

Lemma 8.1. *Repair decreases system entropy but increases environmental entropy by at least W_{repair}/T.*

9 Worked Example: Inletting-driven Expansion

Consider

$$S_{k+1} = (1 + \alpha T_k)S_k, \quad T_k = T_0 + \eta_k.$$

Define

$$E(N) = \sum_{n=0}^{k-1} \frac{\eta_n^2}{2\sigma^2}.$$

This yields Gaussian weights and factorized partition function. Entropy and specific heat can be computed analytically.

10 Simulation Guidelines

1. Generate ensembles of nests with parameters (T_0, r, η).
2. Compute $E(N)$, $Z(\beta)$, $\langle E \rangle$, $S(\beta)$.
3. Plot $C(\beta)$ vs β to detect phase transitions.
4. Simulate operator dynamics as Markov chains, measure $\sigma(t)$.
5. Test fluctuation theorems with forward/reverse protocols.
6. Estimate entropy rate via compression algorithms.

11 Discussion

The UNNS thermodynamic layer provides:

- a statistical mechanics of recursion;
- metrics for complexity and instability;
- bridges to information theory and cognitive science;
- a framework for non-equilibrium recursive processes.

12 Conclusion

UNNS thermodynamics introduces energy, entropy, temperature, and work into recursive dynamics. Phase transitions, fluctuation theorems, and entropy rates reveal deep analogies with statistical mechanics. This positions UNNS as a substrate not only for mathematical structure but also for complex non-equilibrium systems.

References

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